Factor theorem to Determinant :-

In a polynomial (Δ) in x, if Δ vanishes $(\Delta = 0)$ for $x = a$, then $(x-a)$ is factor of Δ

Eg:-

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Put $x = b$ in Δ

$$\Delta = \begin{vmatrix} 1 & b & b^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad R_1 \equiv R_2$$

(Two rows are identical then degree of factor is 1)

$\Delta = 0$

$\therefore (x-b)^1$ is factor of Δ.

Note:-

* If 'r' (rows/columns) are identical in Δ for $x = a$, then $(x-a)^{r-1}$ is a factor of Δ.

Eg:-

$$\Delta = \begin{vmatrix} a & x & x \\ x & a & x \\ x & x & a \end{vmatrix}$$

Put $a = x$ in Δ

$$\Delta = \begin{vmatrix} x & x & x \\ x & x & x \\ x & x & x \end{vmatrix} \quad R_1 \equiv R_2 \equiv R_3$$

(Three rows are identical then degree of factor is 2)

$\Delta = 0$

$\therefore (a-x)^2$ is a factor of Δ

* If determinant is in cyclic symmetric form then $m = \left\{\text{degree of product of leading diagonal}\right\} - \left\{\text{degree of product of factor}\right\}$

✓ If $m=0$, then other factor is k.

✓ If $m=1$, then other factor is $k(a+b+c)$

✓ If $m=2$, then other factor is
$k(a^2+b^2+c^2)+l(ab+bc+ca)$

where k, l are constants.

Ex: 7.3.

1). S.T $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$

Sol

$\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$

$(x-a)^2=0$
$x-a=0$
$x=a.$

$x+2a=0$
$x=-2a$

Put $x=a$ in Δ.

$\Delta = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix}$ $R_1 \equiv R_2 \equiv R_3$

$\Delta = 0$

$\therefore (x-a)^2$ is a factor of Δ.

Put $x=-2a$ in Δ.

$\Delta = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 \\ a & -2a & a \\ a & a & -2a \end{vmatrix}$ $R_1 \to R_1+R_2+R_3$

$\Delta = 0$

$\therefore (x+2a)$ is factor of Δ.

$$M = \left.\begin{array}{c}\text{degree of}\\ \text{Product leading}\\ \text{diagonal}\end{array}\right\} - \begin{array}{c}\text{degree of product}\\ \text{of factor.}\end{array}$$

The degree of product of leading diagonal $x^1 \cdot x^1 \cdot x^1$ is 3

The degree of product of factor $(x-a)^2 \cdot (x+2a)^1$ is 3.

$$M = 3 - 3$$
$$M = 0.$$

If $M=0$ then other factor is K.

$\therefore$ The factors are $K(x-a)^2(x+2a)$

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = K(x-a)^2(x+2a) \longrightarrow (1)$$

$x = 2a$ in (1)

$$\begin{vmatrix} 2a & a & a \\ a & 2a & a \\ a & a & 2a \end{vmatrix} = K(2a-a)^2(2a+2a)$$

$$\cancel{a \times a \times a} \begin{vmatrix} \overset{+}{2} & \overset{-}{1} & \overset{+}{1} \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = K(a)^2(4a)$$

$$2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 4K$$

$$2(4-1) - 1(2-1) + 1(1-2) = 4K$$

$$2(3) - 1(1) + 1(-1) = 4K$$

$$6 - 1 - 1 = 4K$$

$$\cancel{4} = \cancel{4}K \Rightarrow K = \frac{\cancel{4}^{1}}{\cancel{4}}$$

$$\boxed{K = 1}$$

use $K = 1$ in (1)

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = 1(x-a)^2(x+2a)$$

$\underline{\text{H.P}}$

H.p.

5M

2) P.T $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

Sol:-

$a=0$

$$\Delta = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$$

Put $a=0$ in Δ.

$$\Delta = \begin{vmatrix} b+c & 0-c & 0-b \\ b-c & c+0 & b-0 \\ c-b & c-0 & 0+b \end{vmatrix}$$

$$\Delta = \begin{vmatrix} b+c & -c & -b \\ b-c & c & b \\ c-b & c & b \end{vmatrix}$$

$$\Delta = cb \begin{vmatrix} b+c & -1 & -1 \\ b-c & 1 & 1 \\ c-b & 1 & 1 \end{vmatrix} \quad C_2 \equiv C_3$$

$$\Delta = cb(0)$$

$$\Delta = 0$$

a is factor of Δ.

Similarly if $b=0$ & $c=0$, then we get $\Delta = 0$.

$\therefore$ b & c is a factor of Δ.

The degree of product of leading diagonal $(b+c)^1 \cdot (c+a)^1 (a+b)^1$ is 3

The degree of product of factor $a^1 \cdot b^1 \cdot c^1$ is 3.

$m = 3-3 \Rightarrow m=0.$

If $m=0$ then the other factor is k.

$\therefore$ The factors are $kabc$.

$$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = kabc \longrightarrow (1).$$

To find k, put $a=1$, $b=1$, $c=1$,

$$\begin{vmatrix} \overset{+}{2} & \overset{-}{0} & \overset{+}{0} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = k(1)(1)(1)$$

$$2\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0 + 0 = k$$

$$2(4-0) = k \Rightarrow \boxed{k=8}$$

use $k=8$ in (1)

$$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$$

\# H.p.

6). S.T $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Sol:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z \end{vmatrix}$$

$x - y = 0$

$x = y$.

Put $x = y$ in Δ

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ y & y & z \\ y^2 & y^2 & z \end{vmatrix} \quad C_1 \equiv C_2$$

$\Delta = 0$

$\therefore (x-y)$ is a factor of Δ.

III'ly, Put $y = z$ & $z = x$, we get $\Delta = 0$

$\therefore (y-z)$ & $(z-x)$ is a factor of Δ.

The degree of product of leading diagonal $1^0 \cdot y^1 \cdot z^2$ is 3.

The degree of product of factor $(x-y)^1 \cdot (y-z)^1 \cdot (z-x)^1$ is 3.

$m = 3 - 3 \Rightarrow m = 0$

If $m = 0$, then other factor is k.

$\therefore$ The factors are $k(x-y)(y-z)(z-x)$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x) \to (1).$$

To find k put $x=1$, $y=2$, $z=3$.

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = k(1-2)(2-3)(3-1)$$

$$1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = k(-1)(-1)(2)$$

$$1(18-12) - 1(9-3) + 1(4-2) = 2k$$

$$1(6) - 1(6) + (2) = 2k$$

$$\cancel{6} - \cancel{6} + 2 = 2k \Rightarrow 2 = 2k$$

$$k = \frac{2}{2} \Rightarrow \boxed{k=1}$$

use $k=1$ in (1) equ.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = 1(x-y)(y-z)(z-x)$$

H.p