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1. Matrices & Determinants

Defn of a matrix:

A matrix is a rectangular array consist no. of rows & columns within Square bracket (open parenthesis)

Eg:
$$A = \begin{array}{cc} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \end{array} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & 2\times 3 \end{array}$$

order of matrix is defined by rows & columns.

General form of matrix:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

where $m \to$ no. of rows
$n \to$ no. of columns

Types of matrix:

* **Row Matrix:**

It consist only one row.

eg: $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$

* **Column Matrix:**

It consist only one column.

eg: $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

* **Square Matrix:**

no. of rows & no. of columns are equal.

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}_{2\times 2}$

* **Diagonal Matrix:**

A Square matrix is Said to be a Diagonal Matrix if it consist values only in diagonal other entries are zero.

eg: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Upper triangular Matrix:

A Square Matrix is said to be upper triangular matrix if below the leading diagonal values are zero.

eg: $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

Lower triangular Matrix:

A Square Matrix is said to be Lower triangular matrix if above the leading diagonal values are zero.

eg: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

Scalar Matrix:

A diagonal Matrix is said to be Scalar Matrix if all diagonal values are equal except one.

$A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

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Identity Materix:

A diagonal Materix is Said to be Identity Materix it consist value 1 in diagonal.

eg: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero materix:

A materix contains all entries are Zero.

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose Materix: (T)

* Interchange row into column

(or)

* Interchange column into row.

Operations on Matrices:-

$+, -$ → Binary operator

* Addition & Subtraction of Matrices:-

Let A & B be two Matrices.

Then Add or Sub of A & B is denoted by $A+B$ (or) $A-B$ defined by

✓ order of both Matrices Should be equal

✓ Add/sub the corresponding elements.

Eg:- 1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}_{2\times3}$; $B = \begin{bmatrix} 0 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix}_{2\times3}$

Sol

$$A+B = \begin{bmatrix} 1+0 & 2+1 & 3+4 \\ 1+2 & 0-1 & 1+3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 3 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ (-) $B = \begin{bmatrix} 0 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix}$

$$A-B = \begin{bmatrix} 1-0 & 2-1 & 3-4 \\ 1-2 & 0+1 & 1-3 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$$

Product / Multiplication of Matrices:

Let A & B be two Matrices
Then product of A & B is denoted by AB and it is defined by

no. of columns in 1st Matrix = no. of rows in 2nd Matrix

Then only product of Matrix is exist.

eg:- $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

product of AB Matrix is exist. $\boxed{(AB)_{2\times 1}}$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + 2\times 2 + 1\times 3 \\ 3\times 1 + 4\times 2 + 0\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 4 + 3 \\ 3 + 8 + 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 \\ 11 \end{bmatrix}_{2\times 1}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ \hline 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+3 \\ 3+8+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

<u>eg:-</u> $A = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

$$AB = \begin{bmatrix} 1 & 0 & 0 & 1 \\ \hline 1 & 2 & 1 & 2 \\ 4 & 0 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+2 \\ 4+0 & 0+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$

Symmetric Matrix:-

A square Matrix is said to be Symmetric Matrix if $A = A^T$.

Eg:-

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

$$\Rightarrow A^T = A.$$

Skew-Symmetric Matrix:-

A square Matrix is said to be Skew-Symmetric if $A^T = -A$ (or) $A = -A^T$.

Eg:-

$$A = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 6 \\ -4 & -6 & 0 \end{bmatrix}; \quad A^T = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -6 \\ 4 & 6 & 0 \end{bmatrix}$$

$$A^T = -A \text{ (or) } A = -A^T.$$

Equal Matrices:-

Let A & B be two Matrices.

Then A is equal to B (i.e) (A=B) is defined by

(i) Order of Matrix should be equal.

(ii) Corresponding elts should be equal in both Matrices.

eg:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}_{2\times2} ; \quad B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}_{2\times2}$$

$\Rightarrow A = B$

eg 2:

$$A = \begin{bmatrix} 1 & -4 \\ 0 & -2 \end{bmatrix}_{2\times2} \quad B = \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}_{2\times2}$$

$A \neq B$.

Properties on Matrices:-

* Commutative pty:-

$$A + B = B + A$$

$$AB \neq BA \text{ (In general)}$$

* Associative pty:-

$1 + (2+3) = (1+2)+3$

$6 = 6$

$$A + (B + C) = (A + B) + C$$

$$A(BC) = (AB)C$$

* Distributive pty:-

$$A \times (B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

* Identity pty:-

Multiplicative Identity is 1

Additive Identity is 0

$$A + O = O + A = A$$

$1 + 0 = 1$

$x =$

$$AI = IA = A$$

For Matrix

1) Additive Identity is O (zero Matrix)

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2) Multiplicative Identity is

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* Inverse pty:-

$2 + (-2) = 0$

$$A + (-A) = 0$$

$$A \times A^{-1} = I \text{ (or) } A^{-1}A = I.$$

$2 \times \frac{1}{2} = 1$

Pty:

(Transpose of Transpose Matrix A = A Matrix)

✓ (i) $(A^T)^T = A$

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$(A^T)^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\boxed{(A^T)^T = A}$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(A-B)^T = A^T - B^T$

(iv) $(AB)^T = B^T A^T$ (Reversal Law of Transpose)

Ex: 7.1.

1). Construct a $m \times n$ Matrix $A = [a_{ij}]$, where a_{ij} is given by

(i) $a_{ij} = \frac{(i-2j)^2}{2}$ where $m=2$, $n=3$

Sol:-

order of A is 2×3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$a_{ij} = \frac{(i-2j)^2}{2}$$

$$a_{11} = \frac{[1-2(1)]^2}{2} = \frac{(1-2)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$a_{12} = \frac{[1-2(2)]^2}{2} = \frac{(1-4)^2}{2} = \frac{(-3)^2}{2} = \frac{9}{2}$$

$$a_{13} = \frac{[1-2(3)]^2}{2} = \frac{(1-6)^2}{2} = \frac{(-5)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{[2-2(1)]^2}{2} = \frac{(2-2)^2}{2} = \frac{0^2}{2} = 0$$

$$a_{22} = \frac{[2-2(2)]^2}{2} = \frac{(2-4)^2}{2} = \frac{(-2)^2}{2} = \frac{\cancel{4}^2}{\cancel{2}} = 2$$

$$a_{23} = \frac{[2-2(3)]^2}{2} = \frac{(2-6)^2}{2} = \frac{(-4)^2}{2} = \frac{16}{2} = 8$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}$$ #

(ii) $a_{ij} = \frac{|3i-4j|}{4}$ where $m=3;\ n=4.$

Sol

Order of A is $3\times4.$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times4}.$$

$$a_{11} = \frac{|3(1)-4(1)|}{4} = \frac{|3-4|}{4} = \frac{|-1|}{4} = \frac{1}{4}$$

$$a_{12} = \frac{|3(1)-4(2)|}{4} = \frac{|3-8|}{4} = \frac{|-5|}{4} = \frac{5}{4}.$$

$$a_{13} = \frac{|3(1)-4(3)|}{4} = \frac{|3-12|}{4} = \frac{|-9|}{4} = \frac{9}{4}.$$

$$a_{14} = \frac{|3(1)-4(4)|}{4} = \frac{|3-16|}{4} = \frac{|-13|}{4} = \frac{13}{4}.$$

2). Find the values of P, q, r & S if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -11 \end{bmatrix}$$

Sol:

$p^2-1=1$

$p^2=1+1$

$p^2=2$

$\boxed{P=\sqrt{2}}$

$-31-q^3=-4$

$-q^3=-4+31$

$-q^3=27$

$-q^3=3^3$

$-q=3$

$\boxed{q=-3}$

To find r

$r+1=\frac{3}{2}$

$r=\frac{3}{2}-1$

$r=\frac{3-2}{2}$

$\boxed{r=\frac{1}{2}}$

To find s.

$s-1=-11$

$\boxed{s=-11+1}$

3). Determine the value of $x+y$ if

$$\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$$

Sol:

$4x=x+6$

$4x-x=6$

$3x=6$

$x=\frac{6}{3} \Rightarrow \boxed{x=2}$

$5x-7=y$

$5(2)-7=y$

$10-7=y$

$\boxed{y=3}$

$x+y=2+3=5$ #

4). Determine the Matrices A & B if they satisfies

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0.$$

and

$$A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

Sol:-

$$2A - B = -\begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

$$2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} \rightarrow (1)$$

$$A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} \rightarrow (2)$$

From (1) & (2) equ, To eliminate 'A'

(1) 2A

$$2\cancel{A} - B = \begin{bmatrix} -6 & 6 & 0 \\ -4 & -2 & -1 \end{bmatrix} \rightarrow (1)$$

$$(2) \quad \overset{(-)}{\cancel{2A}} \overset{(+)}{-} 4B = \overset{(-)}{\begin{bmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{bmatrix}} \rightarrow (2)$$

$$3B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{bmatrix}$$

$$3B = \begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

$$B = \frac{1}{3} \begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

To eliminate 'B' in (1) & (2) equ.
To find A.

$$(1) \times 2 \quad 4A - 2\cancel{B} = \begin{bmatrix} -12 & 12 & 0 \\ 8 & -4 & -2 \end{bmatrix}$$

$$(2) \quad \overset{(-)}{A} \overset{(+)}{-} \cancel{2B} = \overset{(-)}{\begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}}$$

$$3A = \begin{bmatrix} -12 & 12 & 0 \\ 8 & -4 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$3A = \begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix} \#$$

5). If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ then Compute A^4.

Sol :-

$2^2 = 2 \times 2 = 4.$

$A^4 = A^2 \times A^2$

$A^2 = A \times A.$

$$A^4 = A^2 \times A^2$$

where $A^2 = A \propto A$

$$A^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cccc} \cancel{1} & 0 & a & 1 \\ \hline 1 & a & 1 & a \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1+0 & a+a \\ 0+0 & 0+1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \left[\begin{array}{cccc} 1 & 0 & 2a & 1 \\ \hline 1 & 2a & 1 & 2a \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$A^4 = \begin{bmatrix} 1+0 & 2a+2a \\ 0+0 & 0+1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$$

Note :-

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} ; \quad A^4 = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} ; \quad A^5 = \begin{bmatrix} 1 & 5a \\ 0 & 1 \end{bmatrix}.$$

7). If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that

$(A-2I)(A-3I) = 0$. find the value of 'x'

Sol:

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix}$$

$$(A-2I)(A-3I) = O.$$

$$\begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 1 & -1 & 2 & x-3 \\ \hline +2 & 2 & +2 & 2 \\ -1 & x-2 & -1 & x-2 \end{array}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-2 & +4+2(x-3) \\ -1-1(x-2) & -2+(x-3)(x-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$-1-1(x-2) = 0$$

$$-1-x+2 = 0$$

$$1-x = 0$$

$$\boxed{x=1}$$

8). If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

Do it as H.W

S.T A^2 is a unit Matrix.

T.P: $A^2 = I$.